Graph Theory

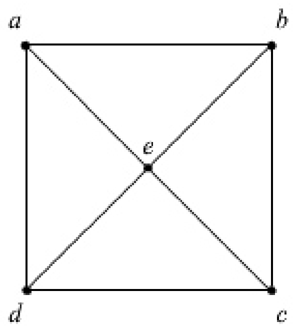
Handshaking Theorem | Paths | Cycles | Adjacency Matrices, with application to counting paths of a given length | Eulerian Graphs | Hamiltonian Graphs | Bipartite Graphs

Terminology (in plain English)

Vertex (plural vertices): The dots in the graphs
Edge: The lines that connect the vertices in the graphs
Simple Graph: a graph that does not have more than one edge connecting the same two vertices.
Multigraph: a graph that has more that one edge connecting the same two vertices (these edges are called multiple or parallel edges)
Undirected Graph: a graph that doesn't have arrows on its edges
Directed Graph: a graph that has arrows on its edges
Adjacent vertices (or neighbor vertices): Two vertices that are connected by an edge.
Incident Edge: an edge is incident to the vertices it connects.
Endpoints of an edge: are the vertices that an edge connects.
Loop: an edge that connects a vertex to its self
Degree of a vertex (for undirected graphs): is the number of edges incident with it, except that a loop contributes twice to the degree of that vertex. The degree of the vertex u is denoted by deg(u)
Isolated Vertex: a vertex that has a degree of 0 (that is not connected to anything).
Pendant Vertex: a vertex that has a degree of 1 (that is connected to only one edge)
Regular Graph: a graph which has only vertices of the same degree
Connected Graph: a graph which has a path joining any two vertices
Tree: a connected undirected graph with no simple circuits.
Isomorphic Graphs: graphs that have the same number of vertices, with the same degrees, the same number of edges, the same edge-vertex relations, the same paths, the same circuits, the same adjacency matrices. However, they may look different.

Example 1

What are the degrees of the vertices in the below graph?



Solution: deg(a) = deg(b) = deg(c) = deg(d) = 3, deg(e) = 4.

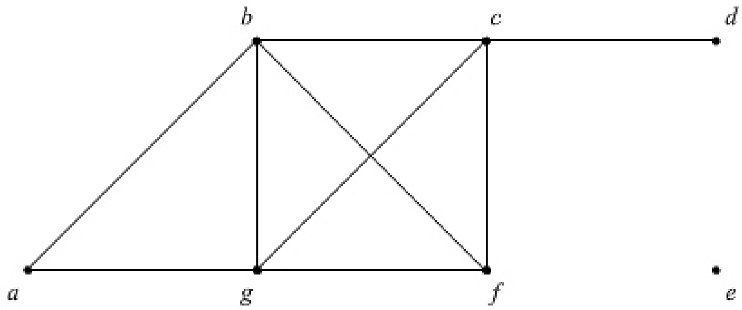
The Handshaking Theorem

The sum of the degrees of all vertices of a graph is twice the number of edges of that graph. (This applies even if multiple edges and loops are present).

$2e = \Sigma \deg(u)$ (where e is the number of edges)

Example 2

How many edges are there in a graph with 1 vertex of degree 2, 3 vertices of degree 4, 1 pendant vertex, 1 isolated vertex, and 1 vertex of degree 3.



Solution: $2e$ = 1x2 + 3x4 + 1x1 + 1x0 + 1x3 = 2 + 12 + 1 + 3 = 18
=> $e = 9$

Paths

A path is a sequence of edges that begins at a vertex of a graph and travels along edges of the graph, always connecting pairs of adjacent vertices.

The path is a *circuit* if it begins and ends in the same vertex.

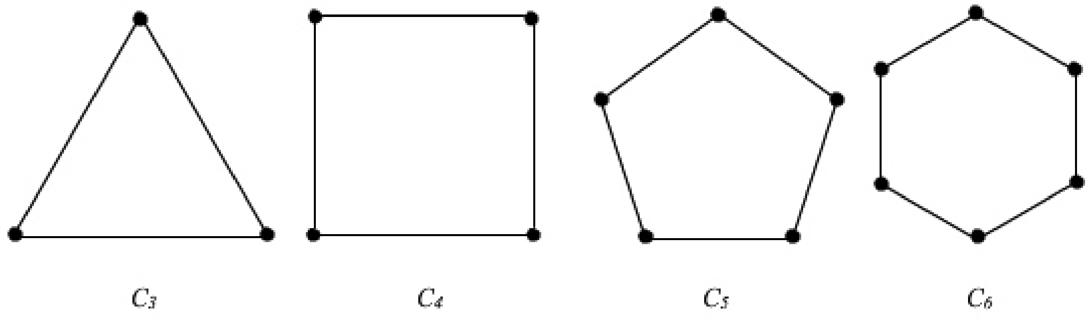
A path or circuit is *simple* if it does not contain the same edge more than once.

The path or circuit is said to *pass through* the vertices $v_1, v_2, v_3, \ldots, v_n$ or *traverse* the edges $e_1, e_2, e_3, \ldots, e_n$.

You describe a path by writing down the vertices it passes through ie. *a, b, c, d, e, f, a*.

Cycles

A cycle graph C_n, n >= 3, consists of n vertices.



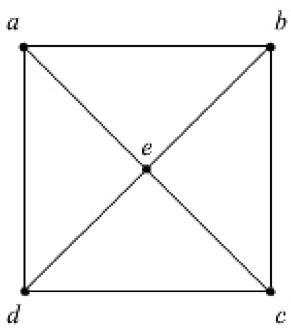
Adjacency Matrices, with application to counting paths of a given length

The simplest way to manipulate graphs in a computer is to store their vertices and edges in lists of edges or adjacency lists. This can become a problem if there is a big number of edges in a graph. To simplify computation, graphs can be represented using matrices. One of the types of matrices one can use is the adjacency matrix.

You can order the vertices of an adjacency matrix however you wish. This means that you can have n! different orderings for a single graph. Usually, to make life easier, we choose to order them alphabetically (*a, b, c, d* …).

The idea is the following: You have a matrix which each column and row is labeled with the vertices. Then you say how many edges connect i with j? And then you put the answer in the ij position of the matrix.

Example 3

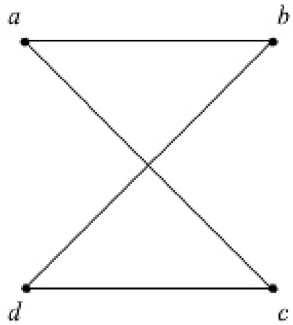


	a	*b*	*c*	*d*	*e*
a	0	1	0	1	1
b	1	0	1	0	1
c	0	1	0	1	1
d	1	0	1	0	1
e	1	1	1	1	0

Adjacency Matrices are cool because you can use them to count the number of paths between vertices. The only thing you need is to know how to multiply tables!

The number of paths of length *r* is calculated by finding the table A^r, where A is your adjacency matrix and by checking the ir value of the A^r matrix.

Example 4



How many paths of length 4 are there from *a* to *d* in the above graph?

Solution: We want to find the value of A^4 in the position $_{1,4}$. So, we calculate

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$A^4 = A \times A \times A \times A = A \times A \times A^2 = A \times A^3$

$$A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

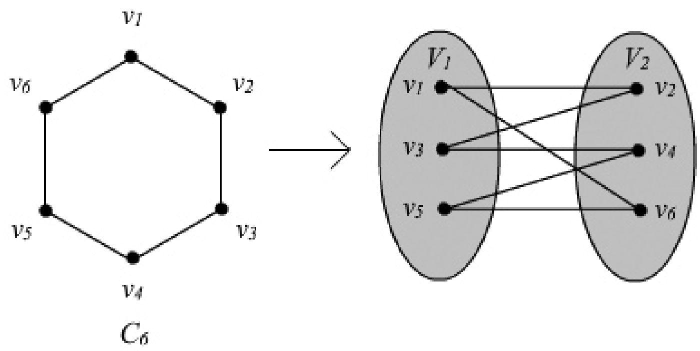
So, as the value in the position in the first row and fourth column is 8 the answer is 8.

In the same fashion if we wanted to find how many paths of length *n* are there from *i* to *j* the we need to calculate A^n and find its *ij* element as the answer.

Bipartite Graphs

A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects to a vertex in V_1 and a vertex in V_2 (so that no edge in the graph connects either two vertices in V_1 or two vertices in V_2). (Not all vertices of V_1 need to be connected with V_2)

Example 5



Show that C_6 is a bipartite graph.

Solution: If you create two subsets V_1 containing v_1, v_3, v_5 and V_2 containing v_2, v_4, v_6 and connect each vertex with its previous and its next (v_{n+1} and v_{n-1}) you get the following diagram which clearly shows that C_6 is a bipartite.

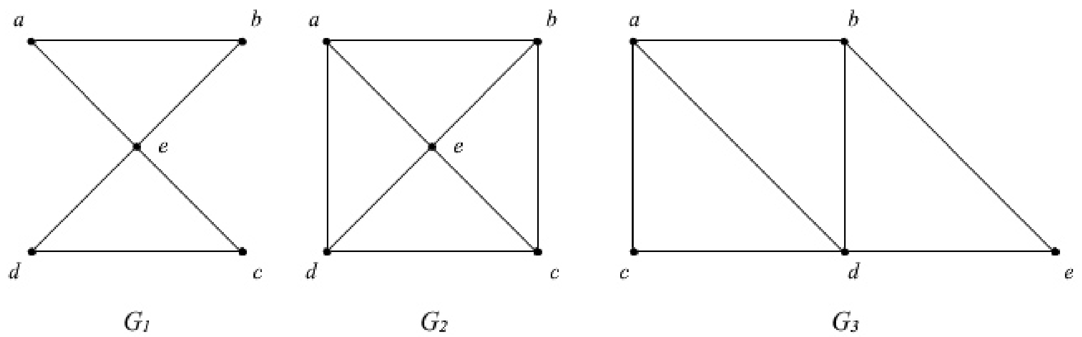
Euler Graphs

Euler Circuit: is a simple circuit containing every edge of the graph.
Euler Path: is a simple path containing every edge of the graph.

A connected multigraph has an Euler circuit if and only if each of its vertices has even degree.

A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Example 6



Which of the above undirected graphs have an Euler circuit? Of those that do not, which have an Euler path?

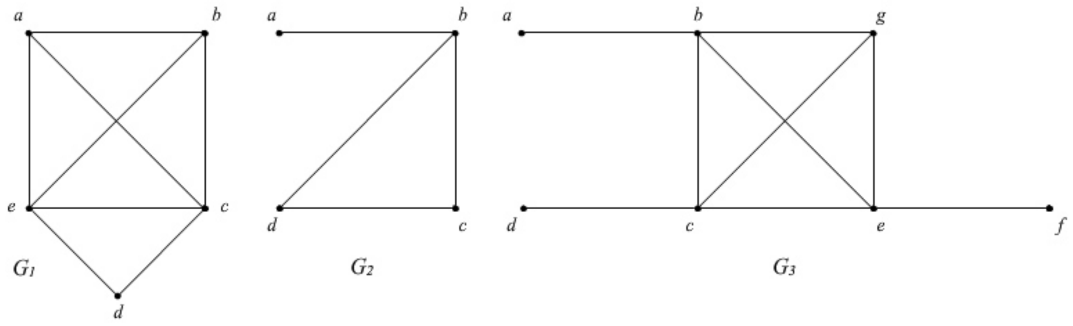
Solution: The graph G_1 has an Euler circuit, ex. a, e, c, d, e, b, a. Non of G_2 and G_3 have an Euler circuit G_2 does not have an Euler circuit because a, b, c, d have odd degree. Similarly, in G_3 a and b have odd degrees as well. As G_3 has only 2 vertices with odd degrees it has an Euler path namely a, c, d, e, b, d, a, b.

Hamilton Graphs

Hamilton Circuit: is a simple circuit that contains every vertex of the graph only once.
Hamilton Path: is a simple path that contains every vertex of the graph only once.
Hamiltonian Graph: is a path that has a Hamiltonian circuit.

Although there is an easy way to determine if a graph has an Euler circuit or path the same does not apply for Hamilton graphs. There are no known simple necessary and sufficient criteria for the existence of Hamilton circuits. However, many theorems are known that give sufficient conditions for the existence of Hamilton circuits. Also, certain properties can be used to show that a graph has no Hamilton circuit.

Example 7



Which of the above simple graphs have a Hamilton circuit or, if not, a Hamilton path?

Solution: G_1 has a Hamilton circuit: a, b, c, d, e, a. There is no Hamilton circuit in G_2 as a is a pendant vertex, but it does have a Hamilton path: a, b, c, d. G_3 has neither a Hamilton circuit or path, since any path containing all vertices must contain one of the edges $\{a, b\}$, $\{e, f\}$, and $\{c, d\}$ more than just once.

Editing / Writing / Graphs:

Vasilis G. Danias

References:

Discreet Mathematics and Its Applications, Kenneth H. Rossen, McGraw-Hill, 5e